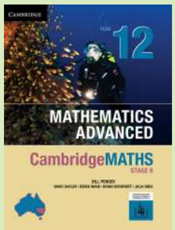
NORMANHURST BOYS HIGH SCHOOL

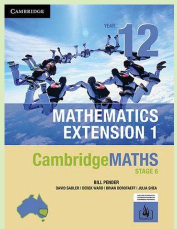
MATHEMATICS ADVANCED
MATHEMATICS EXTENSION 1/2
(YEAR 12 COURSE)

Topic summary and exercises:

A X1 X2 **Calculus with Trigonometric Functions**

With references to





Name: ....

Initial version by H. Lam, February 2014.
Last updated July 23, 2023, with major revision in March 2020 for Mathematics Advanced.
Various corrections by students and members of the Mathematics Department at Normanhurst Boys High School.

Acknowledgements Pictograms in this document are a derivative of the work originally by Freepik at `http://www.flaticon.com`, used under CC BY 2.0.

Symbols used

Beware! Heed warning.

Mathematics Advanced content.

Mathematics Extension 1 content.

Literacy: note new word/phrase.

$\mathbb{R}$ the set of real numbers

$\forall$ for all

Syllabus outcomes addressed

MA12-5 applies the concepts and techniques of periodic functions in the solution of problems involving trigonometric graphs

MA12-6 applies appropriate differentiation methods to solve problems

MA12-7 applies the concepts and techniques of indefinite and definite integrals in the solution of problems

Syllabus subtopics

MA-T3 Trigonometric Functions and Graphs

MA-C2 Differential Calculus

MA-C3 Applications of Differentiation

MA-C4 Integral Calculus

Gentle reminder

- For a thorough understanding of the topic, *every* question in this handout is to be completed!
- Additional questions from *CambridgeMATHS Year 12 Advanced* or *CambridgeMATHS Year 12 Extension 1* will be completed at the discretion of your teacher.
- Remember to copy the question into your exercise book!

Contents

Section 1

Curve sketching and problem solving

Learning Goal(s)

Knowledge
Trigonometric graphs - $\sin x$, $\cos x$ and $\tan x$

Skills
Sketch the trigonometric graphs and their transformations

Understanding
How these graphs can help to assist with problem solving

By the end of this section am I able to:

23.1 Examine and apply transformations to sketch functions of the form $y = kf\Big(a(x+b)\Big) + c$, where a, b, c and k are constants, in a variety of contexts, where $f(x)$ is one of $\sin x$, $\cos x$ or $\tan x$, stating the domain and range when appropriate.

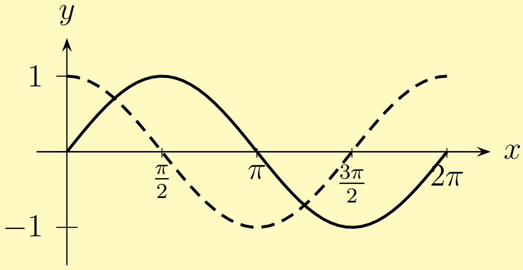
23.2 Solve trigonometric equations involving functions of the form $kf\Big(a(x+b)\Big) + c$, using technology or otherwise, within a specified domain

23.3 Use trigonometric functions of the form $kf\Big(a(x+b)\Big) + c$ to model and/or solve practical problems involving periodic phenomena

1.1 Curve sketching

Laws/Results

Basic $y = \sin x$, $y = \cos x$ curve:



- Domain: $D = \{x : x \in \mathbb{R}\}$
- Range: $R = \{y : -1 \le y \le 1\}$
- Period: $T = \frac{2\pi}{n}$

Transformed curves:

- General equation: $y = a\sin(nx + \phi)$
- a: amplitude
- T: period. (Distance between peaks/troughs)
- n: frequency. (The number of times it "appears" from 0 to 2π)
- ϕ: phase shift

Sketch:

- Always find the period from the frequency

Example 1

Sketch one period of:

1. $y = 2\sin 3x$.

2. $y = 3\cos 4x$.

3. $y = \sin(\pi x)$.

4. $y = 4\sin\frac{2\pi x}{3}$, $x \in [0, 6]$

5. $y = \dfrac{\cos 3\pi x}{4}$

6. $y = 2\cos\frac{3x}{5}$

1.

4.

2.

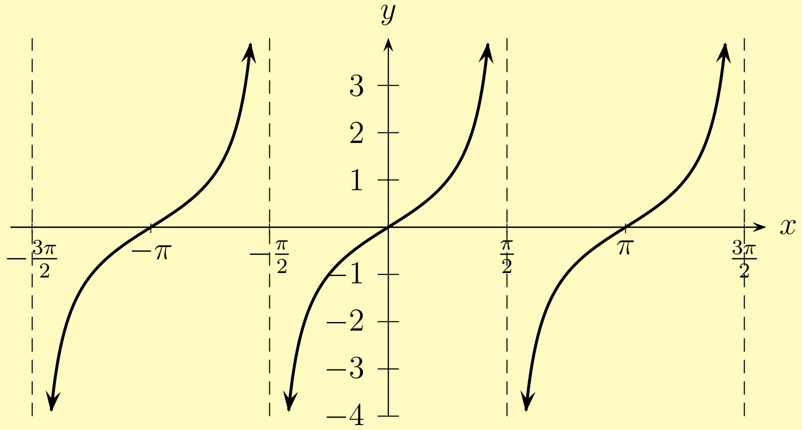
5.

3.

6.

Laws/Results

Basic $y = \tan x$ curve:



- Domain: $D = \{x : x \in \mathbb{R}, x \neq \frac{(2k+1)\pi}{2}\}$
- Range: $R = \{y : y \in \mathbb{R}\}$
- Period: $T = \frac{\pi}{n}$

Transformed curves:

- General equation: $y = a\sin(nx + \phi)$
- a: vertical stretch factor
- T: period. (Distance before graph repeats)
- n: frequency. (The number of times it "appears" from 0 to π)
- ϕ: phase shift

Laws/Results

Basic $y = \operatorname{cosec} x$ curve:

- Domain: $D = \{x : x \in \mathbb{R}, x \neq k\pi\}$
- Range: $R = \{y : y \leq -1 \text{ or } y \geq 1\}$
- Period: $T = \frac{2\pi}{n}$
- Property: odd function

Laws/Results

Basic $y = \sec x$ curve:

- Domain: $D = \{x : x \in \mathbb{R}, x \neq \frac{(2k+1)\pi}{2}\}$
- Range: $R = \{y : y \leq -1 \text{ or } y \geq 1\}$
- Period: $T = \frac{2\pi}{n}$
- Property: even function

Laws/Results

Basic $y = \cot x$ curve:

- Domain: $D = \{x : x \in \mathbb{R}, x \neq k\pi\}$
- Range: $R = \{y : y \in \mathbb{R}\}$
- Period: $T = \frac{\pi}{n}$

Example 2

Sketch the following graphs for $0 \leq x \leq 2\pi$.

1. $y = \sin 2x + 1$

2. $y = \frac{1}{2}\sin\left(x - \frac{\pi}{2}\right) + 3$

3. $y = 4\sin 3\left(x - \frac{\pi}{6}\right) - 2$

4. $y = 2\cos\left(2x + \frac{\pi}{3}\right)$

5. X1 $y = \cos^2 x$

6. X1 $y = \sin x \cos x$

1.

2.

3.

4.

5.

6.

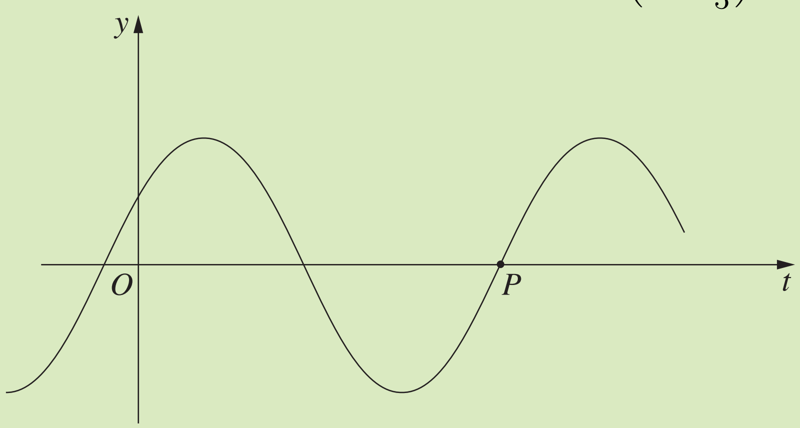
Example 3

[2016 HSC Q8] How many solutions does the equation $|\cos(2x)| = 1$ have for $0 \leq x \leq 2\pi$?

(A) 1 (B) 3 (C) 4 (D) 5

Example 4

[2015 Ext 1 HSC Q10] The graph of the function $y = \cos\left(2t - \frac{\pi}{3}\right)$ is shown below.



What are the coordinates of the point P?

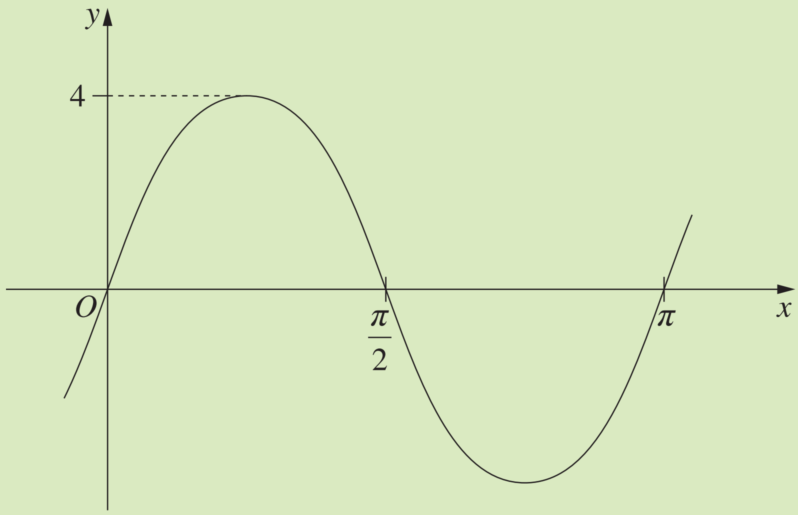
(A) $\left(\frac{5\pi}{12}, 0\right)$ (B) $\left(\frac{2\pi}{3}, 0\right)$ (C) $\left(\frac{11\pi}{12}, 0\right)$ (D) $\left(\frac{7\pi}{6}, 0\right)$

1.2 Problem Solving

1.2.1 Simple graphs

Example 5

[**2010 2U HSC**] The graph shown is $y = A\sin bx$.



i. Write down the value of A. **1**

ii. Find the value of b. **1**

iii. On the same set of axes, draw the graph of $y = 3\sin x + 1$, for $0 \le x \le \pi$. **2**

1.2.2 Transformed graphs

Important note

Look for equivalent π-fraction values along the horizontal axis, and equivalent quadrant cutoff values.

Example 6

[2013 2U HSC Q13] The population of a herd of wild horses is given by

$$P(t) = 400 + 50\cos\left(\frac{\pi}{6}t\right)$$

where t is the time in months.

i. Find all times during the first 12 months when the population equals 375 horses. **2**

ii. Sketch the graph of $P(t)$ for $0 \leq t \leq 12$. **2**

Example 7

[2018 2U HSC Q15] The length of daylight, $L(t)$, is defined as the number of hours from sunrise to sunset, and can be modelled by the equation

$$L(t) = 12 + 2\cos\left(\frac{2\pi t}{366}\right)$$

where t is the number of days after 21 December 2015 for $0 \le t \le 366$.

i. Find the length of daylight on 21 December 2015. **1**

ii. What is the shortest length of daylight? **1**

iii. What are the two values of t for which the length of daylight is 11? **2**

Answer: i. 14 hrs ii. 10 hrs iii. 122/244 days

Example 8

[2009 2U HSC Q7] Between 5 am and 5 pm on 3 March 2009, the height, h, of the tide in a harbour was given by

$$h = 1 + 0.7\sin\frac{\pi}{6}t \quad \text{for } 0 \le t \le 12$$

where h is in metres and t is in hours, with $t = 0$ at 5 am.

i. What is the period of the function h? **1**

ii. What was the value of h at low tide, and at what time did low tide occur? **2**

iii. A ship is able to enter the harbour only if the height of the tide is at least 1.35 m. Find all times between 5 am and 5 pm on 3 March 2009 during which the ship was able to enter the harbour. **3**

Example 9

[2004 Ext 1 HSC Q7] The rise and fall of the tide is assumed to be ~~simple harmonic~~ *modelled by a sinusoidal function*, with the time between successive high tides being 12.5 hours. A ship is to sail from a wharf to the harbour entrance and then out to sea. On the morning the ship is to sail, high tide at the wharf occurs at 2 am. The water depths at the wharf at high tide and low tide are 10 metres and 4 metres respectively.

(i) Show that the water depth, y metres, at the wharf is given by $y = 7 + 3\cos\left(\dfrac{4\pi t}{25}\right)$, where t is the number of hours after high tide. **2**

(ii) An overhead power cable obstructs the ship's exit from the wharf. The ship can only leave if the water depth at the wharf is 8.5 metres or less. **2**

Show that the earliest possible time that the ship can leave the wharf is 4:05 am.

(iii) At the harbour entrance, the difference between the water level at high tide and low tide is also 6 metres. However, tides at the harbour entrance occur 1 hour earlier than at the wharf. In order for the ship to be able to sail through the shallow harbour entrance, the water level must be at least 2 metres above the low tide level. **2**

The ship takes 20 minutes to sail from the wharf to the harbour entrance and it must be out to sea by 7 am. What is the latest time the ship can leave the wharf?

Further exercises

A Ex 2I

- All questions

X1 Ex 3J

- All questions

Section 2

Differentiation of trigonometric functions

2.1 $\sin x$ for small x.

Learning Goal(s)

Knowledge
Approximation of $\sin x$ and $\tan x$ for small values of x

Skills
Operate with the limit $\lim_{x\to 0} \frac{\sin x}{x}$

Understanding
When to use such an approximation

By the end of this section am I able to:

23.2 Establish the limit $\lim_{x\to 0} \frac{\sin x}{x}$.

Laws/Results

If x is small, then

$$\sin x \approx x \approx \tan x$$

Proof

GeoGebra

```
sin x over x.ggb
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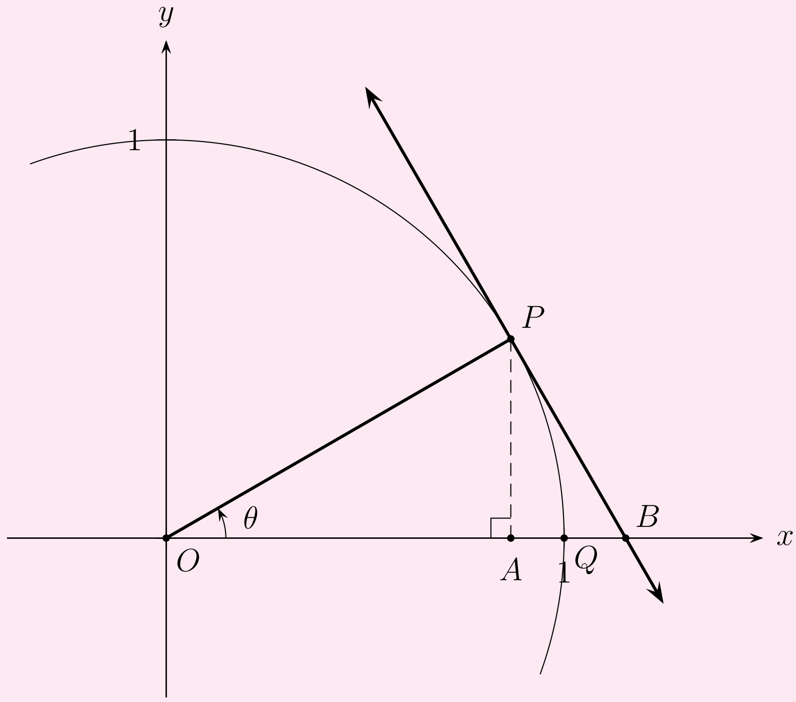
Steps

1. Examine unit circle, with tangent drawn to it at point P.



2. In $\triangle OAP$, $\dfrac{AP}{OP} =$ $\sin\theta$

3. In $\triangle OPB$, $\dfrac{PB}{OP} =$ $\tan\theta$

4. Length of arc PQ: $r\theta = \theta$

5. Comparing lengths,

$$\sin\theta < \theta < \tan\theta$$

6. Hence when θ is small,

$$\sin\theta \approx \theta \approx \tan\theta$$

Example 10

Find the approximate length of the diameter of the moon if it subtends an angle of 31 minutes at an observer on the earth, and the distance of the moon to the earth is 400 000 km.

Answer: 3 607 km

Example 11

X1 If θ is small, show that $\tan\left(\frac{\pi}{4}+\theta\right) \approx \frac{1+\theta}{1-\theta}$.

Example 12

An observer notes that the wing span of an aeroplane directly overhead subtends an angle of 20 minutes at his eye. If the wing span is known to be 30 metres, find the height of the aeroplane to the nearest 100 metres. **Answer:** $\frac{16\,200}{\pi} \approx 5\,200\,\text{m}$

2.2 ⓧ1 $\sin x$ for small x limit

Theorem 1

ⓧ1 Given $\sin x \approx x \approx \tan x$ when x is small, then

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \tag{6.1}$$

$$\lim_{x\to 0} \frac{\tan x}{x} = 1 \tag{6.2}$$

Example 13

Evaluate $\displaystyle\lim_{x\to 0} \frac{\sin 5x}{3x}$.

Answer: $\frac{5}{3}$

Example 14

Evaluate $\displaystyle\lim_{x\to 0} \frac{1-\cos^2 3x}{x^2}$.

Answer: 9

Example 15

Evaluate $\displaystyle\lim_{x\to 0} \frac{\tan 5x}{\sin \frac{1}{3}x}$.

Answer: 15

Example 16

Evaluate $\displaystyle\lim_{x\to 0} \frac{\sin \pi x^\circ}{4x}$.

Answer: $\frac{\pi^2}{720}$

Further exercises

(A) Ex 6A

- Q2, 6, 7

(X1) Ex 7A

- Q2, 6-8

2.3 Derivatives of $\sin x$, $\cos x$ and $\tan x$

Learning Goal(s)

Knowledge
What the derivatives of basic trigonometric functions are

Skills
Differentiate the basic trigonometric functions

Understanding
When to apply the various differentiation rules to curve sketching, optimisation and rates of change problems

By the end of this section am I able to:

23.5 Establish the formulae $\dfrac{d}{dx}(\sin x) = \cos x$ and $\dfrac{d}{dx}(\cos x) = -\sin x$ by numerical estimations of the limits and informal proofs based on geometric constructions.

23.6 Calculate derivatives of trigonometric functions

23.7 Apply the product, quotient and chain rules to differentiate functions of the form $f(x)g(x)$, $\dfrac{f(x)}{g(x)}$ and $f\Big(g(x)\Big)$ where $f(x)$ and $g(x)$ are any of the functions covered in the scope of this syllabus

23.8 Use any of the functions covered in the scope of this syllabus and their derivatives to solve practical and abstract problems

Laws/Results

Derivative of basic trigonometric functions:

$$\frac{d}{dx}(\sin x) = \ldots\cos x\ldots \tag{8.1}$$

$$\frac{d}{dx}(\cos x) = \ldots -\sin x\ldots \tag{8.2}$$

$$\frac{d}{dx}(\tan x) = \ldots\sec^2 x\ldots \tag{8.3}$$

Important note

The ....chain.... ...rule... will be required for most questions.

ICT interactive

Atomi: Visualising derivatives using geometry

2.3.1 (X2) Proof of $\frac{d}{dx}(\sin x) = \cos x$

Steps

1. Given

$$\sin(A+B) = \sin A\cos B + \cos A\sin B \tag{1.1}$$

$$\sin(A-B) = \sin A\cos B - \cos A\sin B \tag{1.2}$$

Subtract (1.2) from (1.1):

$$\sin(A+B) - \sin(A-B) = 2\cos A\sin B \tag{1.3}$$

2. Then, let

$$S = A + B \tag{2.1}$$

$$T = A - B \tag{2.2}$$

Add (2.1) and (2.2) and halving,

$$A = \tfrac{1}{2}(S+T) \tag{2.3}$$

Subtract (2.2) from (2.1) and halving,

$$B = \tfrac{1}{2}(S-T) \tag{2.4}$$

Replace $A+B$ and $A-B$ in (1.3) with (2.1) and (2.2), as well as A and B with (2.3) and (2.4) respectively,

$$\sin S - \sin T = 2\cos\left(\frac{1}{2}(S+T)\right)\sin\left(\frac{1}{2}(S-T)\right) \tag{2.5}$$

3. Now, apply differentiation from first principles on $\sin x$:

$$\begin{aligned}
\frac{d}{dx}(\sin x) &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} && (S = x+h, T = x)\\
&= \lim_{h\to 0}\frac{2\cos\left(\frac{1}{2}(2x+h)\right)\sin\left(\frac{1}{2}h\right)}{h} && (\text{Use (6.1)})\\
&= \cos x
\end{aligned}$$

2.3.2 Standard questions

Example 17

Evaluate $\dfrac{d}{dx}\left(4\sin\left(3x-\frac{\pi}{3}\right)\right)$.

Example 18

Evaluate $\dfrac{d}{dx}\left(\frac{3}{2}\tan\frac{3}{2}x\right)$.

2.3.3 Differentiation rules

Example 19

Differentiate $\tan^2 x$ with respect to x. **Answer:** $2\tan x\sec^2 x$

Example 20

Evaluate $\dfrac{d}{dx}\left(e^x \cos x\right)$.

Answer: $e^x(\cos x - \sin x)$

Example 21

Evaluate $\dfrac{d}{dx}\left(\dfrac{\sin x}{x}\right)$.

Answer: $\frac{x\cos x - \sin x}{x^2}$

Example 22

Evaluate $\dfrac{d}{dx}\left(x^2 \sin\sqrt{x}\right)$

Answer: $\frac{1}{2}x^{\frac{3}{2}}\cos\sqrt{x} + 2x\sin\sqrt{x}$

Further exercises

(A) Ex 6B

- Q1-8, 11-14, 16-17

(x1) Ex 7B

- Q1-8, 11-14, 16-21, 23
- ⚠ Q22, 24

2.4 Applications of differentiation

2.4.1 Equation of tangents and normals

Example 23

[2003 CSSA] Find the equation of the normal to the curve $y = x \sin x$ at the point $x = \frac{\pi}{2}$.

Answer: $x + y - \pi = 0$

2.4.2 Curve sketching

Example 24

[2003 CSSA 2U] Consider the function $f(x) = e^{-x}\cos x$ for $0 \leq x \leq 2\pi$.

(i) Find the values of x where the stationary points occur. **2**

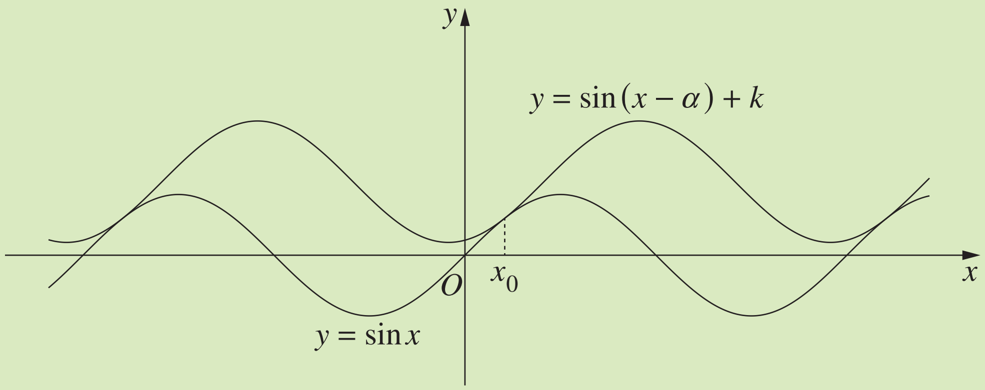
(ii) Determine the nature of the stationary points. **2**

(iii) Sketch the curve showing the coordinates of the stationary points in exact form and the intercepts with the axes. **2**

(iv) Find the number of solutions to the equation $e^{-x}\cos x - \frac{1}{2}x = 0$ in the domain $0 \leq x \leq 2\pi$. Justify your answer. **1**

Example 25

[2019 Ext 1 HSC Q14] X1 The diagram shows the two curves $y = \sin x$ and $y = \sin(x - \alpha) + k$, where $0 < \alpha < \pi$ and $k > 0$. The two curves have a common tangent at x_0, where $0 < x_0 < \frac{\pi}{2}$.



i. Explain why $\cos x_0 = \cos(x_0 - \alpha)$. **1**

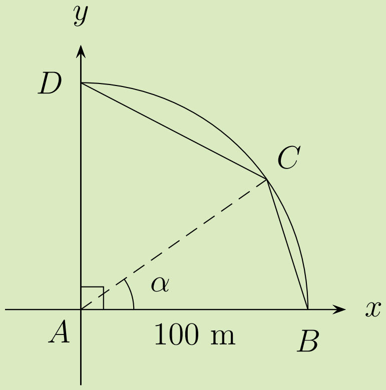
ii. Show that $\sin x_0 = -\sin(x_0 - \alpha)$. **2**

iii. Hence, or otherwise, find k in terms of α. **2**

2.4.3 Minimisation/maximisation

Example 26

[2012 NSBHS Trial] $ABCD$ is a quadrilateral inscribed in a quarter of a circle centred at A with radius 100 m. The points B and D lie on the x and y axes and the point C moves on the circle such that $\angle CAB = \alpha$ as shown in the diagram below.



(i) Solve the equation $\sin(x + 15°) = \cos 24°$. **1**

(ii) Show that the area of the quadrilateral $ABCD$ can be expressed as **3**

$$A = 5\,000(\sin\alpha + \cos\alpha)$$

(iii) Show that the maximum area of this quadrilateral is $5\,000\sqrt{2}$ m^2. **4**

Example 27

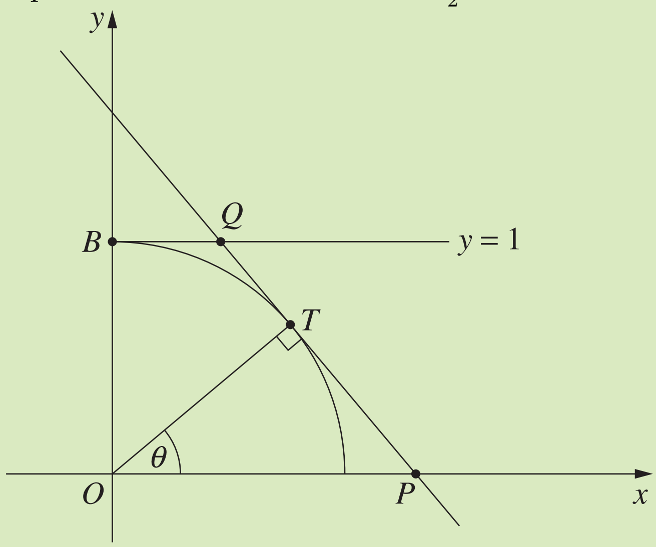
[1998 3U HSC Q7] Consider the function $y = 1 + \sqrt{3}\sin x + \cos x$.

i. Find the equation of the tangent to the graph of the function at $x = \dfrac{5\pi}{6}$. **3**

ii. Find the maximum and minimum values of $1 + \sqrt{3}\sin x + \cos x$ in the interval $0 \le x \le 2\pi$. **2**

Example 28

[2012 2U HSC Q16] The diagram shows a point T on the unit circle $x^2 + y^2 = 1$ at angle θ from the positive x axis where $0 < \theta < \frac{\pi}{2}$.



The tangent to the circle at T is perpendicular to OT, and intersects the x axis at P, and the line $y = 1$ at Q. The line $y = 1$ intersects the y axis at B.

i. Show that the equation of the line PT is **2**

$$x \cos\theta + y \sin\theta = 1$$

ii. Find the length of BQ in terms of θ. **2**

iii. Show that the area, A, of the trapezium $OPQB$ is given by **2**

$$A = \frac{2 - \sin\theta}{2\cos\theta}$$

iv. Find the angle θ that gives the minimum area of the trapezium. **3**

Answer: i. Show ii. $\frac{1-\sin\theta}{\cos\theta}$ iii. Show iv. $\frac{\pi}{6}$

Further exercises

(A) Ex 6C

- Q1, 3 last column
- Q2, 5-16

(X1) Ex 7C

- Q1, 3 last column
- Q2, 5-18, 20

2.4.4 Rates of change/motion

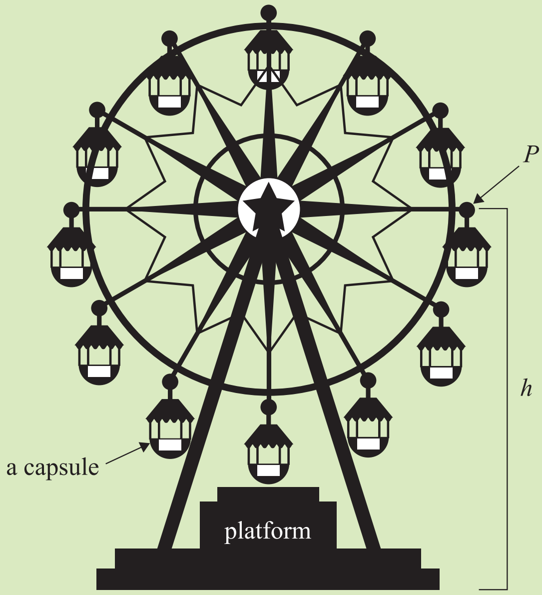
Example 29

[2017 VCE Mathematical Methods Paper 2, Q2] Sammy visits a giant Ferris wheel. Sammy enters a capsule on the Ferris wheel from a platform above the ground. The Ferris wheel is rotating anticlockwise. The capsule is attached to the Ferris wheel at point P. The height of P above the ground, h, is modelled by

$$h(t) = 65 - 55\cos\left(\frac{\pi t}{15}\right)$$

where t is the time in minutes after Sammy enters the capsule and h is measured in metres.

Sammy exits the capsule after one complete rotation of the Ferris wheel.



(a) State the minimum and maximum heights of P above the ground. **1**

(b) For how much time is Sammy in the capsule? **1**

(c) Find the rate of change of h with respect to t and, hence, state the value of t at which the rate of change of h is at its maximum. **2**

Example 30

[1997 2U Q8] A particle is moving along the x axis. Its position at time t is given by

$$x = t + \sin t$$

i. At what times during the period $0 < t < 3\pi$ is the particle stationary? **3**

ii. At what times during the period $0 < t < 3\pi$ is the acceleration equal to 0? **2**

iii. Carefully sketch the graph of $x = t + \sin t$ for $0 < t < 3\pi$. **3**

Clearly label any stationary points and any points of inflection.

Example 31

[2019 Independent 2U Trial Q15] A particle is moving in a straight line. At time t seconds it has displacement $x = e^{-t} \sin t$ metres from a fixed point O on the line and velocity v metres per second.

i. Show that $v = e^{-t} (\cos t - \sin t)$. **1**

ii. Show that successive times at which the particle is at rest forms an arithmetic progression and find its common difference. **2**

iii. Show that the successive displacements of the particle from O when it is at rest form a geometric progression and find its common ratio. **2**

Answer: i. Show ii. $d = \pi$ iii. $r = -e^{-\pi}$

Example 32

[1995 2U HSC Q10]

(a) (i) Draw the graphs of $y = 4\cos x$ and $y = 2 - x$ on the same set of axes for $-2\pi \leq x \leq 2\pi$. **2**

(ii) Explain why all the solutions of the equation $4\cos x = 2 - x$ must lie between $x = -2$ and $x = 6$. **1**

(b) Two particles A and B start moving on the x axis at time $t = 0$. The position of particle A at time t is given by

$$x = -6 + 2t - \frac{1}{2}t^2$$

and the position of particle B is given by

$$x = 4\sin t$$

i. Find expressions for the velocities of the two particles. **2**

ii. Use part (a) to show that there are exactly two occasions, t_1 and t_2, when these particles have the same velocity. **2**

iii. Show that the distance travelled by particle A between these two occasions is **3**

$$4 - 2(t_1 + t_2) + \frac{1}{2}\left({t_1}^2 + {t_2}^2\right)$$

iv. Show that the particles never meet. **2**

Further exercises

Ⓐ Ex 7A

- Q12-13

Ⓐ Ex 7B

- Q5, 9, 14

(X1) Ex 7C

- Q1, 3 last column
- Q2, 5-18, 20

(X1) Ex 9B

- Q6, 11, 12, 15

(X1) Ex 9D

- Q5, ⚠ Q14

Section 3

Integration resulting in trigonometric functions

Learning Goal(s)

Knowledge
What the primitives of basic trigonometric functions are

Skills
Integrate to obtain the basic trigonometric functions

Understanding
When to apply the various integration rules to curve area under the curve, area between the curve, rate of change, motion and the trapezoidal rule problems

By the end of this section am I able to:

23.9 Establish and use the formulae for the anti-derivatives of $\sin(ax+b)$, $\cos(ax+b)$ and $\sec^2(ax+b)$

23.10 Calculate the area under a curve

23.11 Calculate areas between curves determined by any functions within the scope of this syllabus

23.12 Use the Trapezoidal rule to estimate areas under curves

Example 33

Evaluate $\displaystyle\int \cos\frac{1}{3}x\,dx$.

Example 34

Evaluate $\displaystyle\int \left(2\sin 3x + \sec^2(5-3x)\right)\,dx$.

Answer: $-\frac{2}{3}\cos 3x - \frac{1}{3}\tan(5-3x)$

Example 35

Evaluate $\displaystyle\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \left(\sec^2 2x - \cos x\right)\ dx$.

Answer: $\sqrt{3} - 1$

Example 36

Evaluate $\displaystyle\int_0^{\frac{\pi}{12}} \frac{dx}{\cos^2 2x}$.

Answer: $\frac{1}{2\sqrt{3}}$

3.1 Harder integrals

3.1.1 Differentiate, then integrate

- At times, rearrange derivative to obtain integral required.

Example 37

(a) Use the product rule to differentiate $x \sin x$.

(b) Hence evaluate $\displaystyle\int x \cos x \, dx$.

Answer: $x \sin x + \cos x + C$

Example 38

Differentiate $\cos^5 x$, hence evaluate $\displaystyle\int_0^\pi \sin x \cos^4 x \, dx$. **Answer:** $\frac{2}{5}$

3.1.2 Other harder integrals

Important note

- These integrals are not limited to any particular type of function!
- Some questions target X1 trigonometric identities.
- Most of the time the chain rule will be in use.

Example 39

Evaluate $\displaystyle\int \tan x \, dx$.

Example 40

Evaluate $\displaystyle\int \cot 2x \, dx$.

Example 41

(X1) Evaluate $\displaystyle\int \sin 2x \cos 2x \, dx$.

Example 42

Evaluate $\displaystyle\int \tan^2 2x \, dx$.

Example 43

[1999 3U HSC]

(i) By equating coefficients of $\sin x$ and $\cos x$ or otherwise, find the constants A and B satisfying the identity **2**

$$A(2\sin x + \cos x) + B(2\cos x - \sin x) \equiv \sin x + 8\cos x$$

(ii) Hence evaluate $\displaystyle\int \frac{\sin x + 8\cos x}{2\sin x + \cos x} \, dx$. **2**

Example 44

[2006 Sydney Grammar 2U Trial]

(i) Show that $\dfrac{d}{dx}(\sec x) = \sec x \tan x$. **2**

(ii) Show that $\dfrac{d}{dx}(\sec x + \tan x) = \sec x (\sec x + \tan x)$. **1**

(iii) Hence or otherwise, evaluate $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec x \, dx$, leaving your answer in simplest exact form. **3**

Answer: $\ln \frac{2+\sqrt{3}}{\sqrt{2}+1}$

Further exercises

(A) Ex 6D

- Q1-2 last 2 columns
- Q3, 6-17

(X1) Ex 7D

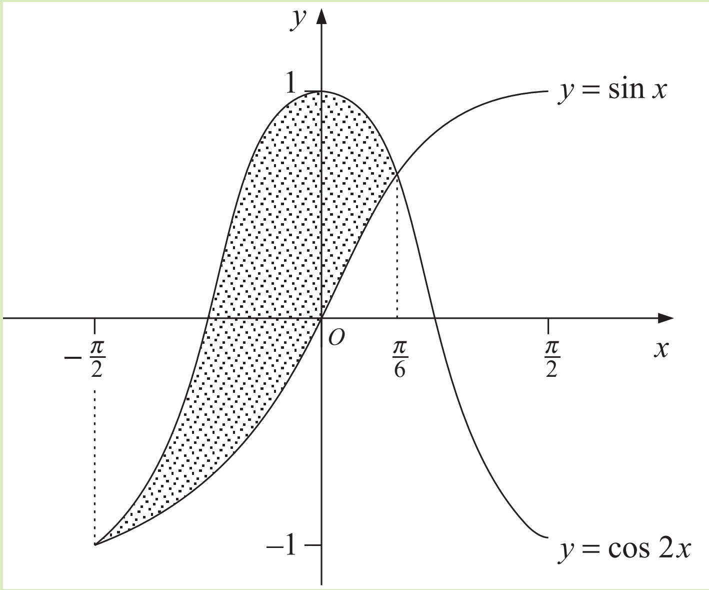
- Q1-2 last column
- Q6-19, 21

3.2 Applications of integration

3.2.1 Area between two curves

Example 45

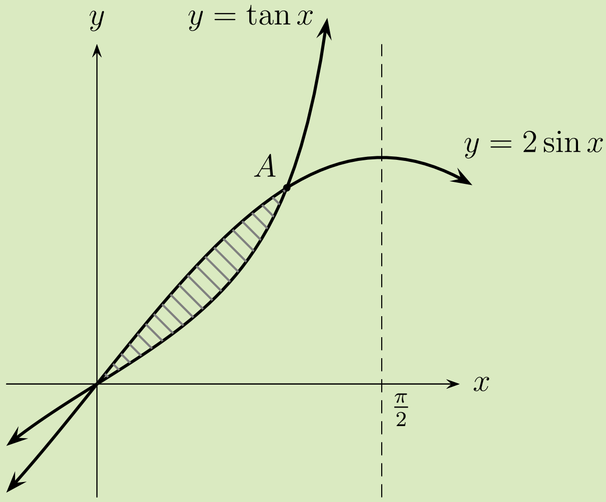
[**1998 2U HSC Q9**] The diagram shows the graphs of the functions $y = \cos 2x$ and $y = \sin x$ between $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$. The two graphs intersect at $x = \frac{\pi}{6}$ and $x = -\frac{\pi}{2}$.



Calculate the area of the shaded region.

Example 46

[2017 NBHS 2U Trial Q12] The diagram below shows the curves $y = \tan x$ and $y = 2\sin x$.



i. Show that the coordinates of the point A are $\left(\frac{\pi}{3}, \sqrt{3}\right)$. **1**

ii. Show that $\dfrac{d}{dx}\left(\log_e \cos x\right) = -\tan x$. **1**

iii. Hence find the area of the shaded region in the diagram. **2**

3.2.2 Trapezoidal Rule

Example 47

[2017 2U HSC Q14] (Modified)

i. Find the exact value of $\displaystyle\int_0^{\frac{2\pi}{3}} \sin x\,dx$. **1**

ii. Using the Trapezoidal Rule with five function values, find an approximation to **2**

$$\int_0^{\frac{2\pi}{3}} \sin x\,dx$$

leaving your answer in terms of π and $\sqrt{3}$.

iii. Using parts (i) and (ii), show that **1**

$$\pi \approx \frac{12}{2+\sqrt{3}}$$

3.2.3 Rates of change

Example 48

[2015 2U Q15] Water is flowing in and out of a rock pool. The volume of water in the pool at time t hours is V litres. The rate of change of the volume is given by

$$\frac{dV}{dt} = 80\sin(0.5t)$$

At time $t = 0$, the volume of water in the pool is 1 200 litres and is increasing.

(i) After what time does the volume of water first start to decrease? **2**

(ii) Find the volume of water in the pool when $t = 3$. **1**

(iii) What is the greatest volume of water in the pool? **1**

Answer: (i) $t = 2\pi$ (ii) 1 348.68 L (iii) 1 520

Example 49

[2006 CSSA Ext 1] X1 At time t minutes the volume flow rate R kilolitres per minute of water into a tank is given by $R = 4\sin^2 t$, $0 \le t \le \pi$.

(i) Find the maximum rate of flow of water into the tank. **1**

(ii) Find the total amount of water which flows into the tank. Give the answer correct to the nearest litre. **2**

Answer: $6.283\,\text{kL}$

3.2.4 Motion

Example 50

[2010 2U HSC Q7] The acceleration of a particle is given by

$$\ddot{x} = 4\cos 2t$$

where x is displacement in metres and t is time in seconds.

Initially the particle is at the origin with a velocity of $1\,\mathrm{ms}^{-1}$.

(i) Show that the velocity of the particle is given by **2**

$$\dot{x} = 2\sin 2t + 1$$

(ii) Find the time when the particle first comes to rest. **2**

(iii) Find the displacement, x, of the particle in terms of t. **2**

Example 51

[2003 2U HSC Q7] The velocity of a particle is given by

$$v = 2 - 4\cos t \qquad \text{for } 0 \leq t \leq 2\pi$$

where v is measured in metres per second and t is measured in seconds.

i. At what times during this period is the particle at rest? **2**

ii. What is the maximum velocity of the particle during this period? **2**

iii. Sketch the graph of v as a function of t for $0 \leq t \leq 2\pi$. **2**

iv. Calculate the total distance travelled by the particle between $t = 0$ and $t = \pi$. **3**

Further exercises

(A) **Ex 6E**

- Q4-16

(A) **Ex 7C**

- Q8 & 9: parts (e), (f)
- ⚠ Q17

(A) **Ex 7C**

- Q13, 14

(X1) **Ex 7E**

- Q2-22

(X1) **Ex 9C**

- Q4(a) & (b): part iii.
- Q13

(X1) **Ex 9F**

- Q7, 10

NESA Reference Sheet – calculus based courses

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}\left[2a + (n - 1)d\right] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1},\ r \neq 1$$

$$S = \frac{a}{1 - r},\ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

$$A = \frac{1}{2}ab\sin C$$

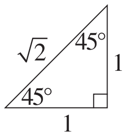
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

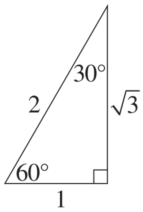
$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$





Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1 + t^2}$

$$\cos A = \frac{1 - t^2}{1 + t^2}$$

$$\tan A = \frac{2t}{1 - t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

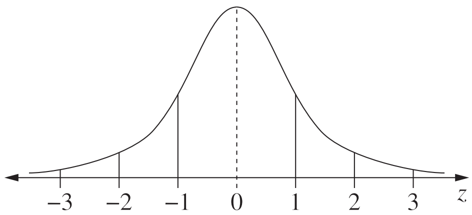
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between −1 and 1
- approximately 95% of scores have z-scores between −2 and 2
- approximately 99.7% of scores have z-scores between −3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \le x) = \int_a^x f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r (1 - p)^{n - r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow \ P(X = x)$$

$$= \binom{n}{x} p^x (1 - p)^{n - x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1 - p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)\left[f(x)\right]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+\left[f(x)\right]^2}$

Integral Calculus

$$\int f'(x)\left[f(x)\right]^n dx = \frac{1}{n+1}\left[f(x)\right]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln\left|f(x)\right| + c$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-\left[f(x)\right]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+\left[f(x)\right]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f\left(x_1\right) + \cdots + f\left(x_{n-1}\right)\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$